ADMM, or the alternating direction method of multipliers, has become the most widely used proximal method in signal processing, but as this study reveals, the method turns out to be a deficient rearrangement of a more practical approach that we propose, which generalizes the Douglas-Rachford algorithm (DR).

Our method stems from a notion of proximity that we develop, which offers new findings on local inversion and infimal postcomposition. Compared to ADMM, our approach enjoys not only a more reasonable form but also a more general convergence result.

**Key result**

The generalized proximal mapping inherits many properties of the vanilla iteration of the possibly non-injective operator. We provide this property in the following result:

**Lemma** In the context of generalized proximity, let \( y \) be a vector in \( F \), and suppose that \( f(x) = 1 + x + y \) is strongly convex.

Then, \( A \) is invertible at any proximal point that is equal to \( y \), which is to say,

\[
\text{prox}_{\gamma f}(x - A y) = x - y,
\]

for any \( \gamma > 0 \) and \( x \) in the null space of \( A \).

**Proximity**

**Context** Fix a real number \( y > 0 \). Let \( f(x) \) be a vector in \( F \), and \( A \) an operator in \( B\mathbb{R}^n;\mathbb{R}^{m} \). Let \( \gamma \) and \( \lambda \) be sequences in \( \mathbb{R} \) such that \( \gamma > 0 \) and \( \lambda \geq 0 \). Suppose that \( f(x) \) is strongly convex.

**Definition** Let \( \gamma > 0 \) and \( \lambda \geq 0 \). The generalized proximity mapping is defined as

\[
\text{prox}_{\gamma f}(x) = \text{arg min}_{y \in F} \left\{ f(y) + \frac{1}{2\gamma} \| x - y \|^2 \right\}.
\]

**Theorem** In the context of generalized proximity, \( (A - \lambda) / \gamma f \) is in \( F \).

Having repeated the operation with \( \gamma, \lambda \) as above, we can then find an unique solution satisfying

\[
\text{prox}_{\gamma f}(x - A y) = x - y,
\]

Not only can \( A \) solve this problem; quite remarkably, \( A \) produces the optimal \( (A - \lambda) / \gamma f \) as a byproduct!

**Proposed algorithm**

**Context** Fix a real number \( y > 0 \). Let \( f(x) \) be a vector in \( F \), and \( A \) an operator in \( B\mathbb{R}^n;\mathbb{R}^{m} \). Let \( \gamma \) and \( \lambda \) be sequences in \( \mathbb{R} \) such that \( \gamma > 0 \) and \( \lambda \geq 0 \).

**Observation** ADMM suffers from a major shortcoming: it requires extra memory, requiring both \( x_n \) and \( z_n \) for the \( x_n \) update under relaxation (\( \lambda_n = 1 \)).

**Conclusion**

Our novel concept of proximity allows us to generalize DR, enhancing it beyond ADMM—both in theory and practice. Using our version of DR, we have recovered ADMM, giving the proximal and general convergence result of our algorithm, and have thus revealed that existing algorithms actually converge more generally than previously reported. In fact, the range of convergence for many versions of ADMM now requires no duality through duality, and the convergence of many versions no longer requires finite dimensions or injective operators.