**Supplementary Material for**

**IN-NETWORK LINEAR REGRESSION WITH ARBITRARILY SPLIT DATA MATRICES**

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**Proof of Proposition 2**

We shall prove the following result:

**Proposition.** Consider a real vector space \( \mathcal{V} \), and let \( v_1, \ldots, v_m \) be vectors in \( \mathcal{V} \). Suppose that \( g: \mathcal{V} \to \mathbb{R} \) is a convex function, and let \( \epsilon \) be a real number. If

\[
    g(v_1 + \cdots + v_m) \leq \epsilon,  \quad (A)
\]

then for all positive numbers \( p_1, \ldots, p_m \) that sum to 1, there exist vectors \( \Delta v_1, \ldots, \Delta v_m \) in \( \mathcal{V} \) that sum to zero such that

\[
    g(p_1^{-1}(v_1 + \Delta v_1)) \leq \epsilon, \quad \ldots, \quad g(p_m^{-1}(v_m + \Delta v_m)) \leq \epsilon.  \quad (B)
\]

If, on the other hand, \( (B) \) holds for some positive numbers \( p_1, \ldots, p_m \) that sum to 1 and some vectors \( \Delta v_1, \ldots, \Delta v_m \) in \( \mathcal{V} \) that sum to zero, then \( (A) \) holds.

**Proof.** First, suppose that \( (A) \) holds. For any positive numbers \( p_1, \ldots, p_m \) that sum to 1, define

\[
    \Delta v_i = p_i(v_1 + \cdots + v_m) - v_i, \quad i = 1, \ldots, m.
\]

Then, not only is it true that \( \Delta v_1 + \cdots + \Delta v_m = 0 \), but since

\[
    g(p_i^{-1}(v_i + \Delta v_i)) = g(p_i^{-1}(v_i + p_i(v_1 + \cdots + v_m) - v_i)) = g(v_1 + \cdots + v_m) \leq \epsilon
\]

for every \( i \), it follows that \( (B) \) holds. Since the positive numbers \( p_1, \ldots, p_m \) that sum to 1 are arbitrary, \( (B) \) holds for all such numbers with the corresponding vectors \( \Delta v_1, \ldots, \Delta v_m \) that sum to zero.

Next, suppose that \( (B) \) holds for some positive numbers \( p_1, \ldots, p_m \) that sum to 1 and some vectors \( \Delta v_1, \ldots, \Delta v_m \) in \( \mathcal{V} \) that sum to zero. Then, by Jensen’s Inequality (due to the convexity of \( g \)),

\[
    g(v_1 + \cdots + v_m) = g\left(\sum_{i=1}^{m} p_i p_i^{-1}(v_i + \Delta v_i)\right) \leq \sum_{i=1}^{m} p_i g(p_i^{-1}(v_i + \Delta v_i)) \leq \epsilon,
\]

and it follows that \( (A) \) holds. \( \square \)

According to the result above, by implying \( (A) \), the assertion that

\( (B) \) holds for some positive numbers \( p_1, \ldots, p_m \) that sum to 1 and some vectors \( \Delta v_1, \ldots, \Delta v_m \) in \( \mathcal{V} \) that sum to zero

implies that

for all positive numbers \( p_1, \ldots, p_m \) that sum to 1, there exist vectors \( \Delta v_1, \ldots, \Delta v_m \) in \( \mathcal{V} \) that sum to zero such that \( (B) \) holds.

Since the converse clearly holds, these statements are, in fact, equivalent. This observation leads to Proposition 2.

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